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# SECOND SCIENTIFIC REPORT

## Realizability Conditions on Two-Port Network Parameters

Robert Maisel

March 1963

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REALIZABILITY CONDITIONS ON TWO-PORT  
NETWORK PARAMETERS

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## ABSTRACT

In this paper, the necessary and sufficient conditions for the realizability of linear, passive, time invariant two-port networks are derived in terms of the various two port parameters. The two-port parameters considered are: the open circuit impedance (Z) parameters, the short circuit admittance (Y) parameters, the hybrid (H) parameters, the transmission (ABCD) parameters, and the scattering (S) parameters. The derivation is based on the definition of a positive real impedance matrix for a two-port network.

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# REALIZABILITY CONDITIONS FOR VARIOUS NETWORK PARAMETERS

## 1. Introduction

The impedance matrix associated with a passive, linear, time invariant, n-port network is positive real. An  $n \times n$  matrix,  $[Z_{ij}(s)]$ , is defined as positive real if and only if it satisfies the following conditions:<sup>1</sup>

$$a) \sum_{i,j=1}^n x_i^* x_j Z'_{ij}(s) \geq 0, \text{ for any } x_i x_j, \text{ and for } \operatorname{Re}(s) > 0,$$

$$\text{where } x_i^* \text{ denotes the complex conjugate of } x_i. \quad (1)$$

$$b) \text{ All } Z_{ij}(s) \text{ are analytic for } \operatorname{Re}(s) > 0, \quad (2)$$

$$c) \text{ All } Z_{ij}(s) \text{ are real for } s \text{ real.} \quad (3)$$

In equation (1)  $Z'_{ij}(s) = \frac{1}{2}[Z_{ij} + Z_{ij}^*]$  are the elements of an Hermitian matrix. The  $Z_{ij}$  are the impedance coefficients of the network, considered as functions of the complex variable  $s = \sigma + j\omega$ , evaluated for  $\operatorname{Re}(s) > 0$ . In matrix form,  $[Z'_{ij}]$  is given by

$$[Z'_{ij}] = \begin{bmatrix} \frac{1}{2}(Z_{11} + Z_{11}^*) & \frac{1}{2}(Z_{12} + Z_{21}^*) & \cdots & \frac{1}{2}(Z_{1n} + Z_{n1}^*) \\ \vdots & & & \\ \frac{1}{2}(Z_{n1} + Z_{1n}^*) & \frac{1}{2}(Z_{n2} + Z_{2n}^*) & \cdots & \frac{1}{2}(Z_{nn} + Z_{nn}^*) \end{bmatrix}$$

A necessary and sufficient condition for inequality (1) to be satisfied is that all the principal determinants of  $[Z'_{ij}]$  are non-negative. The individual terms of the matrix  $[Z'_{ij}]$  are of the form,

$$Z_{11} = r_{11} + jX_{11} ,$$

$$Z_{11}^* = r_{11} - jX_{11} ,$$

etc. Therefore, all elements on the principal diagonal are real, and  $[Z'_{ij}]$  can be expressed as

$$[Z'_{ij}] = \begin{bmatrix} r_{11} & \frac{1}{2}(Z_{12} + Z_{21}^*) & \cdots & \frac{1}{2}(Z_{1n} + Z_{n1}^*) \\ \frac{1}{2}(Z_{21} + Z_{12}^*) & r_{22} & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{2}(Z_{n1} + Z_{1n}^*) & \cdots & \cdots & r_{nn} \end{bmatrix}$$

In addition, if the network is reciprocal, then  $Z_{ij} = Z_{ji}$  and the matrix reduces to

$$[Z'_{ij}] = \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ r_{12} & r_{22} & \cdots & r_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ r_{1n} & r_{2n} & \cdots & r_{nn} \end{bmatrix}$$

## 2. Realizability Conditions on the Immittance Parameters

Consider the passive, linear, time-invariant, two-port network shown in Figure 1.

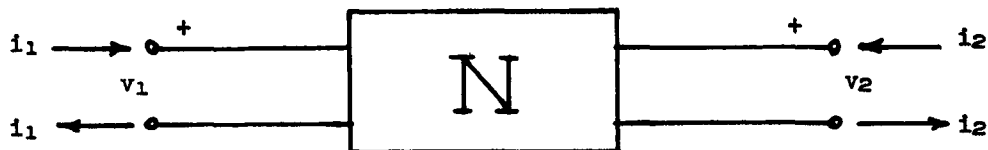


Figure 1

By immittance parameters we shall mean those parameters characterizing the network which can be defined by choosing one variable from terminal-pair #1, one from terminal-pair #2, and by relating these to the remaining two variables. In general, three sets of parameters are defined by this procedure. They are:

a) The "z" parameters:

$$v_1 = z_{11}i_1 + z_{12}i_2$$

$$v_2 = z_{21}i_1 + z_{22}i_2$$

b) The "y" parameters:

$$i_1 = y_{11}v_1 + y_{12}v_2$$

$$i_2 = y_{21}v_1 + y_{22}v_2$$

c) The "h" parameters:

$$v_1 = h_{11}i_1 + h_{12}v_2$$

$$i_2 = h_{21}i_1 + h_{22}v_2$$

Realizability conditions for the  $z$  parameters only may be immediately deduced from a consideration of equations (1), (2), and (3). As has been mentioned, a necessary and sufficient condition for (1) to be satisfied is that all the principal determinants of  $[Z'_{ij}]$  are non-negative, for  $\text{Re}(s) > 0$ . Application of this criterion to the reciprocal case gives (for  $\text{Re}(s) > 0$ ):

$$r_{11} \geq 0 \quad (4a)$$

$$r_{11}r_{22} - r_{12}^2 \geq 0 \quad (4b)$$

which imply

$$r_{22} \geq 0 \quad (4c)$$

These conditions, together with conditions (2) and (3), suffice to assure that the impedance function of the associated network is positive real.

A more general set of conditions, valid for any of the immittance parameters, may be deduced by following a procedure outlined by Stern.<sup>2</sup>

Clearly, conditions (4a), (4b), and (4c) constitute a special case. Stern deduces restrictions on a generalized set of immittance parameters for a two-port network imposed by passivity conditions. That is, he considers networks characterized by equations of the form

$$D_1 = k_{11}J_1 + k_{12}J_2$$

$$D_2 = k_{21}J_1 + k_{22}J_2$$

where the D's and J's are, respectively, currents or voltages, and the subscripts refer to the terminals of the network. The k's may be any of the three immittance parameters. For a two-port network, a necessary and sufficient condition for inequality (1) to be satisfied is that, for  $\text{Re}(s) > 0$ ,

$$\text{Re}(k_{11}) \geq 0 \quad (5a)$$

$$|k_m| + \text{Re}(k_m) \leq 2\text{Re}(k_{11}) \text{Re}(k_{22}) \quad (5b)$$

which imply

$$\text{Re}(k_{22}) \geq 0 \quad (5c)$$

where  $k_m = k_{12}k_{21}$ .

For a reciprocal network,  $k_{12} = k_{21}$ . If the network is to be described by the h parameters, conditions (5) reduce to

$$\text{Re}(h_{11}) \geq 0 ,$$

$$\text{Re}(h_{22}) \geq 0 ,$$

$$|h_{12}|^2 + \text{Re}(h_{12}^2) \leq 2\text{Re}(h_{11}) \text{Re}(h_{22}) ,$$

when  $\text{Re}(s) > 0$ .

Now, if

$$\operatorname{Re}(h_{1j}) = h_{1j} ,$$

$$\operatorname{Im}(h_{1j}) = H_{1j} ,$$

are the real and imaginary parts of  $h_{1j}$ , then the third of the realizability conditions above can be written as,

$$\sqrt{(h_{12}^2 - H_{12}^2)^2 + (2h_{12}H_{12})^2 + (h_{12}^2 - H_{12}^2)} \leq 2h_{11}h_{22} ,$$

which reduces to

$$h_{12}^2 + H_{12}^2 + (h_{12}^2 - H_{12}^2) \leq 2h_{11}h_{22} ,$$

or

$$2h_{12}^2 \leq 2h_{11}h_{22}$$

The three conditions on the  $h$  parameters may therefore be written as

$$h_{11} \geq 0 \quad (6a)$$

$$h_{22} \geq 0 \quad (6b)$$

$$h_{11}h_{22} - h_{12}^2 \geq 0 \quad (6c)$$

for  $\operatorname{Re}(s) > 0$ .

These conditions are identical in form with the conditions (4a), (4b), (4c), on the  $z$  parameters, as expected.

The  $h$  parameters are related to the  $z$  parameters by the relations,

$$[h] = \begin{bmatrix} h_{11}h_{12} & \\ h_{12}h_{22} & \end{bmatrix} = \begin{bmatrix} |z|/z_{22} & z_{12}/z_{22} \\ z_{12}/z_{22} & 1/z_{22} \end{bmatrix},$$

and to the  $y$  parameters by,

$$[h] = \begin{bmatrix} h_{11}h_{12} & \\ h_{12}h_{22} & \end{bmatrix} = \begin{bmatrix} 1/y_{11} & -y_{12}/y_{11} \\ -y_{12}/y_{11} & |y|/y_{11} \end{bmatrix}.$$

Conditions on analyticity, for the  $h$  parameters for  $\text{Re}(s) > 0$  must now be examined. Since  $z_{22} = 1/h_{22}$  may have no poles or zeros in the right-half  $s$ -plane,  $h_{22}$  may have no poles or zeros there. Similarly,  $y_{11} = 1/h_{11}$  implies that  $h_{11}$  has no poles or zeros for  $\text{Re}(s) > 0$ . Finally, since  $y_{12} = h_{12}/h_{11}$  may have zeros but may not have poles, identical conditions apply to  $h_{12}$ . The conditions may be summarized as follows:

- a)  $h_{11}$  and  $h_{22}$  may have no poles or zeros for  $\text{Re}(s) > 0$ .

(6d)

- b)  $h_{12}$  may have zeros but may not have poles for  $\text{Re}(s) > 0$ .

(6e)

Since the elements of the impedance matrix are real for  $s$  real,  $h_{22} = 1/z_{22}$  must be real for  $s$  real; similarly,  $z_{11} = h_{11}/h_{22}$ , which requires that  $h_{11}$  be real for  $s$  real.

From  $y_{12} = -h_{12}/h_{11}$ , a similar condition is deduced for  $h_{12}$ . In summary, it may be stated that  $h$  parameters are real for real values of  $s$ . (6f)

Conditions (6d), (6e), and (6f) together with conditions (6a), (6b), and (6c) are necessary and sufficient to insure that the impedance matrix of the associated network is positive real.

A complete test to establish the positive real character of the impedance matrix of the associated network includes a demonstration of inequalities (6a), (6b), and (6c) over the entire right-half  $s$ -plane. More useful conditions may be deduced, however, in which the inequalities need only be verified on the  $j\omega$  axis.

It is known that

$$H(s) = h_{11}x_1^2 + 2h_{12}x_1x_2 + h_{22}x_2^2$$

is positive real for all values of  $x$ . From the theorem which states that the real part of a function of a complex variable changes sign  $2n$  times in the neighborhood of a pole of multiplicity  $n$ , it can be shown that any pole of  $H(s)$  on the  $j\omega$  axis must be simple, and that any residue at this pole must be real and positive.

Let  $k_{11}$ ,  $k_{22}$ ,  $k_{12}$  be the residues of  $h_{11}$ ,  $h_{22}$ ,  $h_{12}$ , respectively, at a pole on the  $j\omega$  axis, and let  $K$  be the residue of  $H(s)$  at this pole. It is known that  $K$  must be real and positive since  $H(s)$  is positive-real. Therefore,

$$K = k_{11}x_1^2 + 2k_{12}x_1x_2 + k_{22}x_2^2 \geq 0 ,$$

which states that the matrix of residues of the  $h$  parameters, at poles on the  $j\omega$  axis, is positive semi-definite. That is,

$$k_{11} \geq 0$$

$$k_{22} \geq 0$$

$$k_{11}k_{22} - k_{12}^2 \geq 0$$

By applying the maximum modulus theorem to  $H(s)$  over the closed right-half  $s$  plane, excluding poles on the  $j\omega$  axis, it can be demonstrated that  $|H(s)|$  attains its maximum value on the axis.

Therefore, inequalities (6a), (6b), and (6c) need only be verified on the  $j\omega$  axis to assure that they are satisfied in the entire right-half  $s$ -plane. The conditions may then be written:

$$h_{11}(j\omega) \geq 0 \tag{7a}$$

$$h_{22}(j\omega) \geq 0 \tag{7b}$$

$$h_{11}(j\omega)h_{22}(j\omega) - [h_{12}(j\omega)]^2 \geq 0 \tag{7c}$$

All poles on the  $j\omega$  axis are simple, with residues given by:

$$k_{11} \geq 0 \quad (7d)$$

$$k_{22} \geq 0 \quad (7e)$$

$$k_{11}k_{22} - k_{12}^2 \geq 0 \quad (7f)$$

Conditions (7a) to (7f), together with conditions (6d), (6e), and (6f), suffice to assure that the impedance matrix of the associated network is positive real.

### 3. The Chain Parameters

The chain parameters are defined, for the network considered in Figure 1, by the equations

$$e_1 = Ae_2 - B i_2$$

$$i_1 = Ce_2 - D i_2 ,$$

which express input variables in terms of output variables.

These ABCD parameters are related to the  $z$  and  $y$  parameters by the relations,

$$[ABCD] = \begin{bmatrix} A & -B \\ C & -D \end{bmatrix} = \begin{bmatrix} z_{11}/z_{12} & -|z|/z_{12} \\ 1/z_{12} & -z_{22}/z_{12} \end{bmatrix}$$

and

$$[ABCD] = \begin{bmatrix} -y_{22}/y_{12} & 1/y_{12} \\ -|y|/y_{12} & -y_{11}/y_{22} \end{bmatrix}$$

The inverse transformations give,

$$[z] = \begin{bmatrix} z_{11} & z_{12} \\ z_{12} & z_{22} \end{bmatrix} = \begin{bmatrix} A/C & 1/C \\ 1/C & D/C \end{bmatrix},$$

and

$$[y] = \begin{bmatrix} y_{11} & y_{12} \\ y_{12} & y_{22} \end{bmatrix} = \begin{bmatrix} D/B & -1/B \\ -1/B & A/B \end{bmatrix}$$

From equation (1), in terms of the  $z$  parameters, the condition that the principal minors be non-negative in the half-plane  $\text{Re}(s) > 0$  requires that

$$\text{Re}(A/C) \geq 0$$

$$\text{Re}(D/C) \geq 0$$

$$\text{Re}(A/C)\text{Re}(D/C) - [\text{Re}(1/C)]^2 \geq 0$$

Then, if

$$A = a + jA'$$

$$B = b + j\beta$$

$$C = \xi + j\psi$$

$$D = \delta + j\Delta ,$$

the first condition becomes

$$\operatorname{Re}(A/C) = \frac{a\xi + A'\psi}{\xi^2 + \psi^2} \geq 0$$

And, since

$$\begin{aligned} \xi^2 + \psi^2 \geq 0, \quad a\xi + A'\psi \geq 0 \\ (\operatorname{Re}(s) > 0) \end{aligned} \tag{8a}$$

The second condition implies,

$$\operatorname{Re}(D/C) = \operatorname{Re} \left[ \frac{\delta + j\Delta}{\xi + j\psi} \right] \geq 0 ,$$

which requires,

$$\frac{\delta\xi + \Delta\psi}{\xi^2 + \psi^2} \geq 0$$

or

$$\delta\xi + \Delta\psi \geq 0 \quad (\operatorname{Re}(s) > 0). \tag{8b}$$

The final condition implies,

$$\operatorname{Re}(A/C)\operatorname{Re}(D/C) - [\operatorname{Re}(1/C)]^2 \geq 0 ,$$

or

$$\frac{a\xi + A'\psi}{\xi^2 + \psi^2} \cdot \frac{\delta\xi + \Delta\psi}{\xi^2 + \psi^2} - [\operatorname{Re}(1/C)]^2 \geq 0 .$$

Since

$$[\operatorname{Re}(1/C)]^2 = \left[ \operatorname{Re} \frac{1}{\xi + j\psi} \right]^2 = \frac{\xi^2}{(\xi^2 + \psi^2)^2} ,$$

$$\frac{(a\xi + A'\psi)(\delta\xi + \Delta\psi)}{(\xi^2 + \psi^2)^2} - \frac{\xi^2}{(\xi^2 + \psi^2)^2} \geq 0 ,$$

which requires

$$(a\xi + A'\psi)(\delta\xi + \Delta\psi) - \xi^2 \geq 0 \quad (\operatorname{Re}(s) > 0) \quad (8c)$$

An alternate set of conditions, derived by relating the ABCD matrix to the y matrix, may be found. Since the y matrix is positive real, the conditions become:

$$\operatorname{Re}(D/B) \geq 0$$

$$\operatorname{Re}(A/B) \geq 0$$

$$\operatorname{Re}(D/B)\operatorname{Re}(A/B) - [\operatorname{Re}(1/B)]^2 \geq 0$$

The first condition may be written

$$\operatorname{Re}(D/B) = \operatorname{Re} \left( \frac{\delta + j\Delta}{b + j\beta} \right) = \frac{\delta b + \Delta\beta}{b^2 + \beta^2} ,$$

or

$$\delta b + \Delta \beta \geq 0 \quad (\operatorname{Re}(s) > 0) \quad (8a')$$

The second condition requires

$$\operatorname{Re}(A/B) = \frac{1}{b^2 + \beta^2} (ab + A'\beta) \geq 0 ,$$

or

$$ab + A'\beta \geq 0 \quad (\operatorname{Re}(s) > 0) \quad (8b')$$

The last condition implies,

$$\frac{(\delta b + \Delta \beta)(ab + A'\beta)}{(b^2 + \beta^2)^2} - [\operatorname{Re}(1/B)]^2 \geq 0 .$$

Since

$$\operatorname{Re}(1/B) = \operatorname{Re} \left[ \frac{1}{b + j\beta} \right] = \frac{b}{b^2 + \beta^2} ,$$

$$(\delta b + \Delta \beta)(ab + A'\beta) - b^2 \geq 0 \quad (\operatorname{Re}(s) > 0) \quad (8c')$$

Equations (8) and (8') may be simplified by noting that a reciprocal network satisfies the condition:

$$AD - BC = 1$$

The real and imaginary parts of the reciprocity relation give

$$a\delta - A'\Delta - b\xi + \beta\psi = 1 ,$$

and

$$a\Delta + A\delta - b\psi - \beta\xi = 0 .$$

Equations (8) then reduce to:

$$a\xi + A\psi \geq 0 , \quad (9a)$$

$$\delta\xi + \Delta\psi \geq 0 , \quad (9b)$$

$$b\xi + A\Delta \geq 0 , \quad (9c)$$

and equations (8') become

$$\delta b + \Delta\beta \geq 0 , \quad (9a')$$

$$a b + A\beta \geq 0 , \quad (9b')$$

$$b\xi + A\Delta \geq 0 , \quad (9c')$$

when  $\text{Re}(s) > 0$  .

Conditions (9) or (9') are entirely equivalent to inequality (1), for a reciprocal network described by the "chain" parameters.

Conditions on the analyticity of A, B, C, and D must be examined if  $\text{Re}(s) > 0$ . It is known that  $z_{11}$ ,  $z_{22}$ ,  $y_{11}$ ,  $y_{22}$  have no poles or zeros for  $\text{Re}(s) > 0$  since they are positive real; it is also known that  $z_{12}$  and  $y_{12}$  have no poles, but may have zeros, in this region.

Therefore,  $C = 1/z_{12}$  may either be analytic for  $\text{Re}(s) > 0$ , or it may have a pole; it may not have a zero. Since  $A/C = z_{11}$  has no poles or zeros in the right-half  $s$ -plane, any pole of  $A$  must be a pole of  $C$ ; and if  $C$  has no poles or zeros, then  $A$  can have no poles. Additionally, any common poles must be of the same multiplicity.

Finally,  $-B = 1/y_{12}$  may be either analytic for  $\text{Re}(s) > 0$ , or it may have a pole; it may not have a zero, etc. Application of these properties of the  $z$  and  $y$  parameters leads to the general result:

- a)  $A, B, C, D$  may have no zeros for  $\text{Re}(s) > 0$ . (9d)
- b) Either  $A, B, C, D$  have no poles or they have the same poles of the same multiplicity, for  $\text{Re}(s) > 0$ . (9e)

It is required that the elements of the impedance matrix must be real for real values of  $s$ . In particular, this implies that  $1/C$  and, therefore,  $C$  itself must be real for real values of  $s$ . This, in turn, implies the same conditions on  $A, B, D$ . That is,

$$A, B, C, D \text{ are real for } s \text{ real.} \quad (9f)$$

Conditions (9d), (9e), (9f), together with either (9a), (9b), (9c), or (9a'), (9b'), (9c') are necessary and sufficient to insure that the impedance matrix of the associated 2-port, reciprocal network is positive real.

#### 4. The Scattering Parameters

Realizability conditions for networks may also be described in terms of the scattering parameters. Such a procedure is outlined by Carlin.<sup>3</sup>

At a given pair of terminals in a network, "reflected" and "incident" voltages may be defined in terms of terminal voltage and current,  $v$  and  $i$ , by,

$$v^{(i)} = \frac{1}{2} \left[ \frac{v}{\sqrt{Z_c}} + \sqrt{Z_c} i \right]$$

$$v^{(r)} = \frac{1}{2} \left[ \frac{v}{\sqrt{Z_c}} - \sqrt{Z_c} i \right] ,$$

where  $\sqrt{Z_c}$  is a normalizing constant.

For an  $n$ -port network, the scattering co-efficients are defined by the equations:

$$v_1^{(r)} = s_{11}v_1^{(i)} + s_{12}v_2^{(i)} + \dots + s_{1n}v_n^{(i)}$$

$$v_2^{(r)} = s_{21}v_1^{(i)} + s_{22}v_2^{(i)} + \dots + s_{2n}v_n^{(i)}$$

$$\vdots$$

$$v_n^{(r)} = s_{n1}v_1^{(i)} + s_{n2}v_2^{(i)} + \dots + s_{nn}v_n^{(i)}$$

The scattering matrix may be expressed in terms of the impedances or admittances observed at the terminals. The derivation can be developed from the expressions for the normalized

"reflected" and "incident" quantities to give

$$S = (Z + 1)(Z - 1)^{-1} = (1 - Y)(1 + Y)^{-1}$$

where the capital letters indicate matrix quantities.

The condition of reciprocity requires that the  $Z$  and  $Y$  matrices must be symmetrical, and consequently  $S$  must be symmetrical.

On the basis of energy relations and other restrictions, Carlin deduces necessary and sufficient conditions for realizability. For the case of an  $n$ -port reciprocal network, these conditions may be stated as:

- a)  $S(p)$ , the scattering matrix, contains real, rational functions  $S_{ij}(p)$ , analytic for  $\sigma > 0$ . (10)
- b)  $Q = [1 - S^*(p)S(p)]$  is a positive Hermitian matrix for  $\sigma = 0$ , (10a)

where  $p$  is the complex variable,  $p = \sigma + j\omega$ .

For the two-port reciprocal case, the scattering matrix is,

$$S = \begin{bmatrix} s_{11}(p) & s_{12}(p) \\ s_{12}(p) & s_{22}(p) \end{bmatrix}$$

and

$$S^* = \begin{bmatrix} s_{11}^*(p) & s_{12}^*(p) \\ s_{12}^*(p) & s_{22}^*(p) \end{bmatrix},$$

whence

$$S^*(p)S(p) = \begin{bmatrix} (s_{11}s_{11}^* + s_{12}s_{12}^*)(s_{11}^*s_{12} + s_{12}^*s_{22}) \\ (s_{11}s_{12}^* + s_{12}s_{22}^*)(s_{12}^*s_{12} + s_{22}^*s_{22}) \end{bmatrix}$$

and

$$Q = (1 - S^*S) = \begin{bmatrix} 1 - |s_{11}|^2 - |s_{12}|^2 & -(s_{11}^*s_{12} + s_{12}^*s_{22}) \\ -(s_{11}s_{12}^* + s_{12}s_{22}^*) & 1 - |s_{22}|^2 - |s_{12}|^2 \end{bmatrix}$$

Since  $Q = 1 - S^*S$  must be a positive Hermitian matrix, condition (1a) becomes, for  $p = j\omega$ ,

$$\operatorname{Re}(q_{11}) \geq 0, \quad \text{or} \quad (1 - |s_{11}|^2 - |s_{12}|^2) \geq 0 \quad \text{--- (11)}$$

$$\operatorname{Re}(q_{22}) \geq 0, \quad \text{or} \quad (1 - |s_{22}|^2 - |s_{12}|^2) \geq 0 \quad \text{--- (12)}$$

and,

$$\operatorname{Re}(q_{11})\operatorname{Re}(q_{22}) - \operatorname{Re}(q_{12})\operatorname{Re}(q_{12}^*) \geq 0,$$

or

$$\begin{aligned} & (1 - |s_{11}|^2 - |s_{12}|^2)(1 - |s_{22}|^2 - |s_{12}|^2) \\ & - \operatorname{Re}(s_{11}s_{12}^* + s_{12}s_{22}^*)\operatorname{Re}(s_{11}^*s_{12} + s_{12}^*s_{22}) \geq 0 \quad \text{--- (13)} \end{aligned}$$

Conditions (10), (11), (12), and (13) are necessary and sufficient to insure that the impedance matrix of the associated 2-port, reciprocal network is positive real.

### 5. An Example of Pole-Zero Locations

A certain class of networks satisfying the analyticity conditions is examined below for purposes of illustration. Consider the symmetrical lattice network shown in Figure 2.

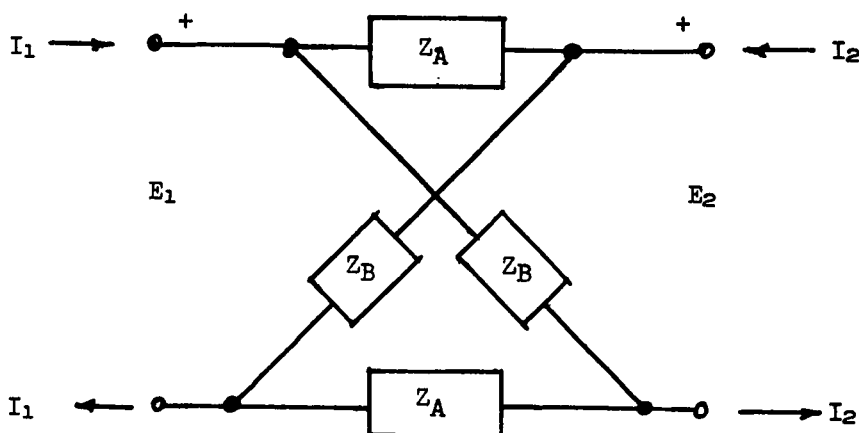


Figure 2

The impedance parameters are

$$z_{11} = z_{22} = \frac{1}{2}(Z_A + Z_B)$$

$$z_{12} = \frac{1}{2}(Z_B - Z_A)$$

In particular, consider the network wherein

$$Z_B = 2R$$

$$Z_A = 2sL ,$$

as in Figure 3:

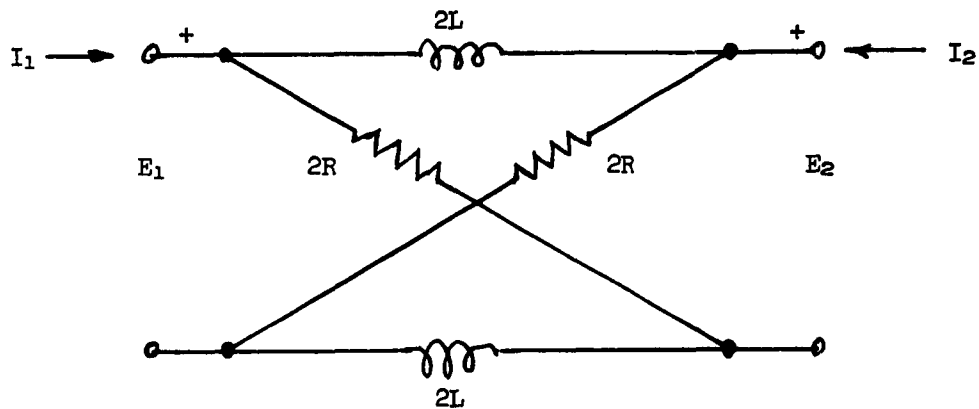


Figure 3

The impedance parameters become

$$z_{11} = z_{22} = L(s + R/L)$$

$$z_{12} = L(s - R/L)$$

The pole-zero locations are shown in Figure 4.

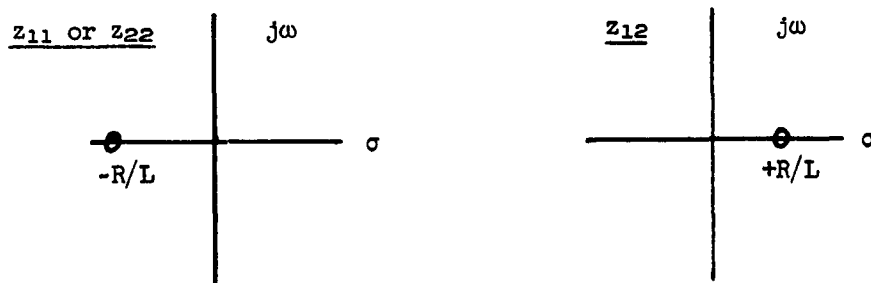


Figure 4

The "chain" parameters are:

$$A = \frac{z_{11}}{z_{12}} = \frac{L(s + R/L)}{L(s - R/L)} = \frac{(s + R/L)}{(s - R/L)}$$

$$B = \frac{|z|}{z_{12}} = \frac{L^2(s + R/L)^2 - L^2(s - R/L)^2}{L(s - R/L)} = \frac{4Rs}{(s - R/L)}$$

$$C = \frac{1}{z_{12}} = \frac{1}{L(s - R/L)}$$

$$D = \frac{z_{22}}{z_{12}} = \frac{(s + R/L)}{(s - R/L)}$$

The pole-zero locations for these parameters are shown in Figure 5.

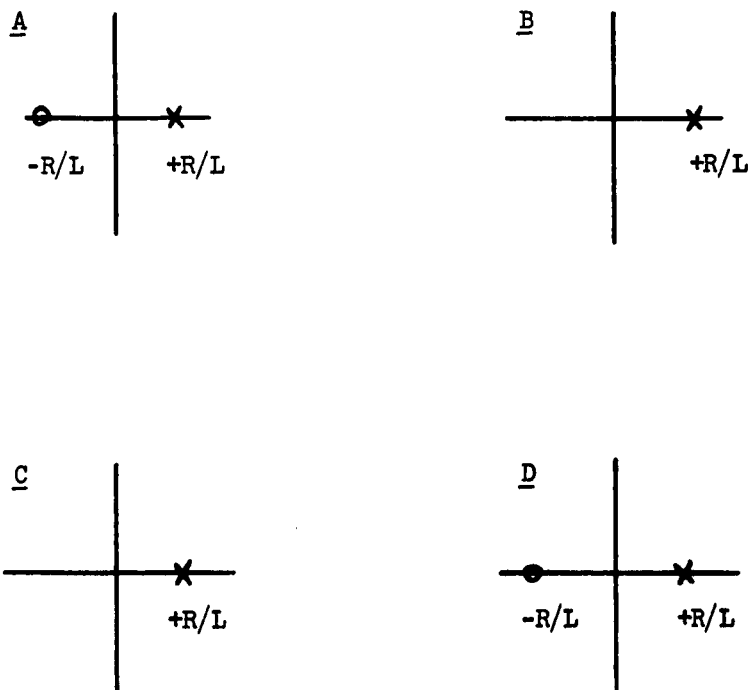


Figure 5

Clearly, these locations satisfy realizability conditions (9d) and (9e).

Similarly, the h parameters are found to be

$$h_{11} = \frac{|z|}{z_{22}} = \frac{4Rs}{(s + R/L)}$$

$$h_{22} = \frac{1}{z_{22}} = \frac{1}{L(s + R/L)}$$

$$h_{12} = \frac{z_{12}}{z_{22}} = \frac{(s - R/L)}{(s + R/L)}$$

The corresponding pole-zero locations are shown in Figure

6.

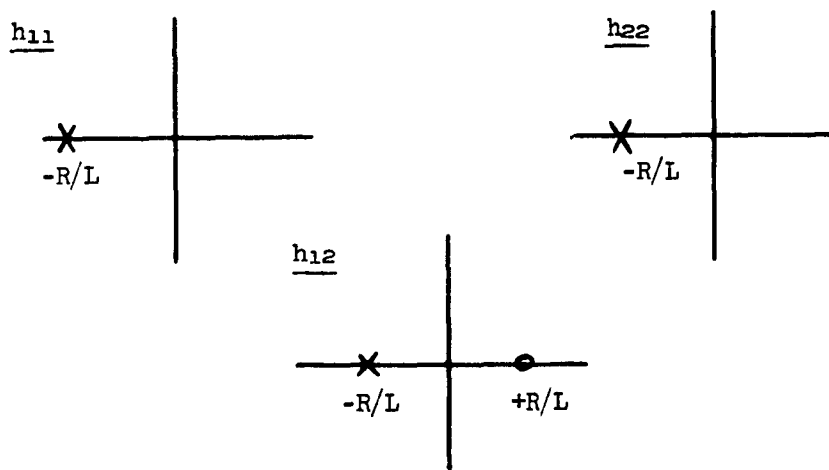


Figure 6

Again, the pole-zero locations are seen to satisfy conditions (6d) and (6e).

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